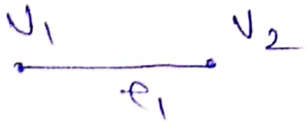
Theorems on Trees

Thm → A tree with n vertices has $(n-1)$ edges.

Pf → We shall prove it by induction on the number of vertices.

If $n=1$ then tree has only one vertex & no edge.

If $n=2$ then tree has two vertices & one edge as shown in figure —



∴ the theorem is true for $n=1$ & $n=2$

We assume that the theorem is true for all trees having less than n vertices.

Let us consider a tree G with 'n' vertices. Then G is a connected graph with no circuit.

Let e_k be any edge in G with end vertices v_i & v_j. Then e_k is the only path b/w v_i & v_j else G will have a circuit.

∴ deletion of e_k from G will disconnect the graph G. ∴ $G-e_k$ is not connected

∴ $G-e_k$ will contain exactly two components

Let these components be G_1 & G_2. Then G_1 & G_2 are also trees as these are circuitless.

Let n_1 & n_2 be no. of vertices in G_1 & G_2 respectively. Then $n_1 < n$ & $n_2 < n$.

∴ No. of edges in $G_1 = n_1 - 1$

& " " " in $G_2 = n_2 - 1$

$\therefore$ no. of edges in $G - e_K = (n_1 - 1) + (n_2 - 1)$

$= n_1 + n_2 - 2$

$= n - 2$, as $n_1 + n_2 = n$

$\therefore$ No. of edges in $G = (n-2) +$ edge e_K

$= n - 2 + 1$

$= n - 1$

(Proved)

Th' Every connected graph with 'n' vertices and $(n-1)$ edges is a tree.

Pf → Let G be a connected graph with n vertices and $(n-1)$ edges. To show that G is a tree, we shall show that G has no circuit. We shall prove it by contradiction. Let G has a circuit. Then removing an edge from circuit will not disconnect G. We remove each edge from all the circuits in G and get a graph G^* such that G^* has no circuit & is connected since removal of edge (from circuit) does not remove vertices, $\therefore$ G^* will contain 'n' vertices

$\therefore$ G^* is a tree with n vertices

Hence G^* has $(n-1)$ edges. But then G will have more than $(n-1)$ edges, which is a contradiction. Hence G has no circuit & is a tree.

(Proved)

Theo. A graph G with n vertices, $(n-1)$ edges & no circuits is a tree

Pf: Let G be a graph with n vertices, $(n-1)$ edges & no circuit. Now if we show that G is connected then G will be a tree. We shall show it by contradiction. We assume that G is disconnected. Then G will have two or more components and every component will be circuit less as G has no circuit. Let G consist of two components say G_1 & G_2. We add an edge 'e' b/w a vertex v_1 in G_1 and v_2 in G_2. This addition of 'e' will not create any circuit because v_1 & V_2 are in different components of G and so there is no path b/w v_1 & v_2. Therefore $G \cup e$ will become a connected graph which has no circuit. Thus $G \cup e$ will be a tree with n vertices & n edges. But we know that a tree with 'n' vertices has $(n-1)$ edges. Thus we get a contradiction. Hence G must be connected.

(Proved)

Thm A graph G is a tree iff it is minimally connected.

Pf→ Let G be a tree. To show that G is minimally connected.
Since G is a tree, it is connected.
If G is not minimally connected than we can remove an edge, say e from G such that G-e remains connected. But then 'e' is in some circuit which implies that G has a circuit. In that case G will not be a tree. which is a contradiction. ∴ G must be minimally connected.

Converse: Let G be minimally connected graph. To show that G is a tree.
Since G is minimally connected, ∴ it is connected. G can not have a circuit because in that case we can remove an edge from the circuit and still leave the graph connected. ∴ G must be a tree.

(Proved)